

# Connected Hypergraphs with Small Spectral Radius

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## Abstract

In 1970 Smith classified all connected graphs with the spectral radius at most 2. Here the spectral radius of a graph is the largest eigenvalue of its adjacency matrix. Recently, the definition of spectral radius has been extended to  $r$ -uniform hypergraphs. In this paper, we generalize the Smith's theorem to  $r$ -uniform hypergraphs. We show that the smallest limit point of the spectral radii of connected  $r$ -uniform hypergraphs is  $\rho_r = (r-1)! \sqrt[r]{4}$ . We discovered a novel method for computing the spectral radius of hypergraphs, and classified all connected  $r$ -uniform hypergraphs with spectral radius at most  $\rho_r$ .

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## 1 Introduction

The spectral radius  $\rho(G)$  of a graph  $G$  is the largest eigenvalue of its adjacency matrix. The connected graphs with spectral radius at most 2 are classified by Smith [23] in 1970: the graphs with spectral radius less than 2 are exactly the simple-laced Dynkin Diagrams:  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$ , and  $E_8$ , while the graphs with spectral radius 2 are the extended simple-laced Dynkin Diagram:  $\tilde{A}_n$ ,  $\tilde{D}_n$ ,  $\tilde{E}_6$ ,  $\tilde{E}_7$ ,  $\tilde{E}_8$ . The simple-laced Dynkin Diagrams have connections to several mathematical fields including Lie groups, Lie algebras, Coxeter groups.

The number 2 is the smallest limit point of the spectral radius of connected graphs. Another important limit point is  $\sqrt{2 + \sqrt{5}} \approx 2.0582$ . Smith and Hoffman [9, 8] developed several important tools to study the spectral radii of graphs. Shearer [22] proved that for any  $\lambda \geq \sqrt{2 + \sqrt{5}}$  there exists a sequence of graphs  $\{G_n\}$  such that  $\lim_{n \rightarrow \infty} \rho(G_n) = \lambda$ . Cvetković et al. [6] gave a nearly complete description of all graphs  $G$  with  $2 < \rho(G) < \sqrt{2 + \sqrt{5}}$ . Their description was completed by Brouwer and Neumaier [1]. Wang et al. [25] studied some graphs with spectral radii close to  $\frac{3}{2}\sqrt{2}$ . Woo-Neumaier [26] and Lan-Lu [11] studied the structures of the connected graphs  $G$  with  $\sqrt{2 + \sqrt{5}} < \rho(G) < \frac{3}{2}\sqrt{2}$ . A *minimizer* graph, denoted by  $G_{n,D}^{min}$ , is a graph which has the minimal spectral radius among all connected graphs of order  $n$  and diameter  $D$ . The problem of determining the minimizer graph is well-studied in the literature [4, 7, 12, 24].

In this paper, we will generalize Smith's theorem to  $r$ -uniform hypergraphs. An  $r$ -uniform hypergraph  $H = (V, E)$  consists of a vertex set  $V$  and an edge-set  $E \subseteq \binom{V}{r}$ . There are roughly two approaches to generalize the spectral theory to  $r$ -uniform hypergraphs. The first approach is to generalize the Laplacian spectra based on the  $s$ -th-order random walks (Rodríguez [20, 20] for  $s = 1$ ,

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Chung [3] for  $s = r - 1$ , and Lu-Peng [14, 15] for general  $1 \leq s \leq r - 1$ .) The second approach is to generalize the spectra of the adjacency matrices base on the Raileigh principle of extremal eigenvalues (for example, Lim [13], Qi [18, 19], Cooper-Dutle [5], Keevash-Lenz-Mubayi [10], and Nikiforov [17], etc.) Let's use the notion of [10, 17]. Given a hypergraph  $H$ , the polynomial form  $P_H(\mathbf{x}): \mathbb{R}^n \rightarrow \mathbb{R}$  is defined for any vector  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  as

$$P_H(\mathbf{x}) = r! \sum_{\{i_1, \dots, i_r\}} x_{i_1} \cdots x_{i_r}.$$

The spectral radius of  $H$ , denoted by  $\rho(H)$ , is defined to be the maximum value of the polynomial form over the  $r$ -norm unit sphere:

$$\rho(H) = \max_{\|\mathbf{x}\|_r=1} P_H(\mathbf{x}).$$

This definition lies in the common interest of [13, 18, 19, 5, 10, 17]. It is a natural generalization of the spectral radius of graphs to hypergraphs. (Noticing in Cooper-Dutle's paper [5], it is off by a constant factor  $(r - 1)!$ . This is not essential and will not affect our classification.)

The number 2 is the spectral radius of the infinite path. (To avoid the definition of the spectral radius of an infinite graph, we really mean that  $2 = \lim_{n \rightarrow \infty} \rho(A_n)$ , where  $A_n$  is the path with  $n$  edges.) For  $r \geq 2$ , let  $\rho_r := (r - 1)! \sqrt[r]{4}$ . It turns out that  $\rho_r = \lim_{n \rightarrow \infty} \rho(A_n^{(r)})$ , where  $A_n^{(r)}$  is the  $r$ -uniform simple path with  $n$  edges. (Here "simple" means that each pair of edges can only intersect at most one vertex.) In this paper, we classified all  $r$ -uniform hypergraphs with spectral radius at most  $\rho_r$ : Theorem 1 and 2 classify all 3-uniform hypergraphs with spectral radius equal to  $\rho_3$  and less than  $\rho_3$ ; Theorem 4 and 5 classify all  $r$ -uniform hypergraphs with spectral radius less than  $\rho_r$  and equal to  $\rho_r$  for all  $r \geq 4$ . These are the most natural generalization of Smith's theorem into  $r$ -uniform hypergraphs.

Our method is different from the method used in Smith's original proof. We actually discovered an easy way to compute the spectral radius using weighted incident matrix. Our method naturally applies to the case  $r = 2$ . Thus, we give another proof for Smith's theorem.

The paper is organized as follows. In Section 2, we introduce the notation and proved several important lemmas for computing the spectral radius. In Section 3, we classify all connected 3-uniform hypergraphs with the spectral radius at most  $\rho_3 = 2\sqrt[3]{4}$ . In Section 4, we introduce the methods of reduction and extension and use them to classify all connected  $r$ -uniform hypergraphs with the spectral radius at most  $\rho_r = (r - 1)! \sqrt[r]{4}$ .

## 2 Notation and Lemmas

An  $r$ -uniform hypergraph  $H$  is a pair  $(V, E)$  where  $V$  is the set of vertices and  $E \subset \binom{V}{r}$  is the set of edges. The degree of vertex  $v$ , denoted by  $d_v$ , is the number of edges incident to  $v$ . If  $d_v = 1$ , we say  $v$  is a leaf vertex. A *walk* on hypergraph  $H$  is a sequence of vertices and edges:  $v_0 e_1 v_1 e_2 \dots v_l$  satisfying that both  $v_{i-1}$  and  $v_i$  are incident to  $e_i$  for  $1 \leq i \leq l$ . The vertices  $v_0$  and  $v_l$  are called the ends of the walk. The length of a walk is the number of edges on the walk. A walk is called a *path* if all vertices and edges on the walk are distinct. The walk is *closed* if  $v_l = v_0$ . A closed walk is called a *cycle* if all vertices and edges in the walk are distinct. A hypergraph  $H$  is called *connected* if for any pair of vertex  $(u, v)$ , there is a path connecting  $u$  and  $v$ . A hypergraph  $H$  is called a *hypertree* if it is connected, and acyclic. A hypergraph  $H$  is called *simple* if every pair of edges intersects at most one vertex. In fact, any non-simple hypergraph contains at least a 2-cycle:  $v_1 F_1 v_2 F_2 v_1$ , i.e.,  $v_1, v_2 \in F_1 \cap F_2$ . A hypertree is always simple.

Now we review the spectral analysis for hypergraphs using the approach of the polynomial form.

**Definition 1.** [2, 5, 10, 17] Given an  $r$ -uniform hypergraph  $H$ , the polynomial form of  $H$  is a function  $P_H(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$  defined for any vector  $\mathbf{x} := (x_1, \dots, x_n) \in \mathbb{R}^n$  as

$$P_H(\mathbf{x}) = r! \sum_{\{i_1, i_2, \dots, i_r\} \in E(H)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

For any  $p \geq 1$ , the largest  $p$ -eigenvalue of  $H$  is defined as

$$\lambda_p(H) = \max_{|\mathbf{x}|_p=1} P_H(x).$$

In this paper, we define the spectral radius of an  $r$ -uniform hypergraph  $H$  to be  $\rho(H) = \lambda_r(H)$ . Equivalently, we have

$$\rho(H) = r! \max_{\substack{\mathbf{x} \in \mathbb{R}_{\geq 0}^n \\ \mathbf{x} \neq 0}} \frac{\sum_{\{i_1, i_2, \dots, i_r\} \in E(H)} x_{i_1} x_{i_2} \cdots x_{i_r}}{\sum_{i=1}^n x_i^r}. \quad (1)$$

Here  $\mathbb{R}_{\geq 0}^n$  denote the closed orthant in  $\mathbb{R}^n$  while  $\mathbb{R}_{> 0}^n$  denote the open orthant. The fraction in Equation (1) is called the *Raileigh quotient*. A non-zero vector  $\mathbf{x}$  maximizing the Raileigh quotient is called an eigenvector corresponding to  $\rho(H)$ . If  $\mathbf{x}$  is an eigenvector, so is  $c\mathbf{x}$  for any scale  $c > 0$ . If an eigenvector  $\mathbf{x}$  has all positive entries, i.e.,  $\mathbf{x} \in \mathbb{R}_{> 0}^n$ , then  $\mathbf{x}$  is called a Perron-Frobenius vector for  $H$ .

**Lemma 1.** [5, 10, 17] If  $H$  is a connected  $r$ -uniform hypergraph, then the Perron-Frobenius vector exists for  $H$ .

By the Lagrange multiplier method, the Perron-Frobenius vector  $\mathbf{x}$  satisfies for any vertex  $v$

$$(r-1)! \sum_{\{v, i_2, \dots, i_r\} \in E(H)} x_{i_2} \cdots x_{i_r} = \rho(H) x_v^{r-1}. \quad (2)$$

We have the following important lemma as a corollary of Lemma 1.

**Lemma 2.** [5, 10, 17] If  $G$  is a connected  $r$ -uniform hypergraph, and  $H$  is a proper subgraph of  $G$ , then

$$\rho(H) < \rho(G).$$

**Definition 2.** A weighted incidence matrix  $B$  of a hypergraph  $H$  is a  $|V| \times |E|$  matrix such that for any vertex  $v$  and any edge  $e$ , the entry  $B(v, e) > 0$  if  $v \in e$  and  $B(v, e) = 0$  if  $v \notin e$ .

**Definition 3.** A hypergraph  $H$  is called  $\alpha$ -normal if there exists a weighted incidence matrix  $B$  satisfying

1.  $\sum_{e: v \in e} B(v, e) = 1$ , for any  $v \in V(H)$ .
2.  $\prod_{v \in e} B(v, e) = \alpha$ , for any  $e \in E(H)$ .

Moreover, the incidence matrix  $B$  is called consistent if for any cycle  $v_0 e_1 v_1 e_2 \dots v_l (v_l = v_0)$

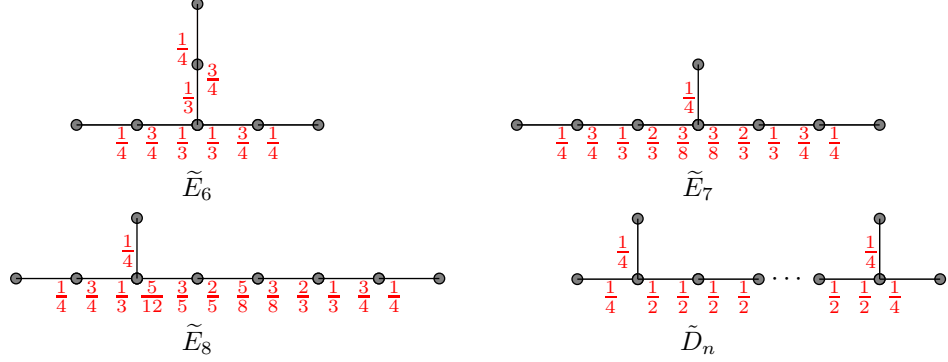
$$\prod_{i=1}^l \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1.$$

In this case, we call  $H$  consistently  $\alpha$ -normal.

**Example 1.** Consider the cycle  $C_n$ . We can define  $B(v, e) = \frac{1}{2}$  for any  $v \in e$ . So  $C_n$  is consistently  $\frac{1}{4}$ -normal.

When  $H$  is a hypertree, any incidence matrix  $B$  of  $H$  is automatically consistent. Here are some examples of  $\frac{1}{4}$ -normal 2-graphs.

**Example 2.** The following graphs:  $\tilde{D}_n$ ,  $\tilde{E}_6$ ,  $\tilde{E}_7$ , and  $\tilde{E}_8$ , are all  $\frac{1}{4}$ -normal. We can show this by labeling the value  $B(v, e)$  at vertex  $v$  near the side of edge  $e$ . If  $v$  is a leaf vertex, then it has the trivial value 1, and we will omit its labeling.



We observe that all connected graphs with spectral radius 2 are consistently  $\frac{1}{4}$ -normal. The relation between the consistent  $\alpha$ -normal labelling and the spectral radius is characterized by the following Lemma.

**Lemma 3.** Let  $H$  be a connected  $r$ -uniform hypergraph. Then the spectral radius of  $H$  is  $\rho(H)$  if and only if  $H$  is consistently  $\alpha$ -normal with  $\alpha = ((r-1)!/\rho(H))^r$ .

*Proof.* We first show that it is necessary. Let  $x := (x_1, \dots, x_n)$  be the Perron-Frobenis eigenvector of  $H$ . Define the weighted incidence matrix  $B$  as follows:

$$B(v, e) = \begin{cases} \frac{(r-1)! \prod_{u \in e} x_u}{\rho(H) x_v^r} & \text{if } v \in e \\ 0 & \text{otherwise.} \end{cases}$$

From this definition, for any edge  $e$ , we have

$$\prod_{v \in e} B(v, e) = \prod_{v \in e} \frac{(r-1)! \prod_{u \in e} x_u}{\rho(H) x_v^r} = \left( \frac{(r-1)!}{\rho(H)} \right)^r = \alpha.$$

Item 2 of Definition 3 is verified. Now we check item 1: for any  $v$ ,  $\sum_e B(v, e) = 1$ .

Recall that the Perron-Fronbenis eigenvector  $\mathbf{x}$  satisfies Equation (2). For any  $v \in V$ , we have

$$\sum_e B(v, e) = \sum_{\{v, i_2, \dots, i_r\} \in E(H)} \frac{(r-1)! \prod_{u \in e} x_u}{\rho(H) x_v^r} = \frac{\rho(H)}{\rho(H)} = 1.$$

To show that  $B$  is consistent, for any cycle  $v_0 e_1 v_1 e_2 \dots v_l (v_l = v_0)$ , we have

$$\prod_{i=1}^l \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = \prod_{i=1}^l \frac{x_{v_{i-1}}^r}{x_{v_i}^r} = 1.$$

Now we show that it is also sufficient. Assume that  $B$  is a consistently  $\alpha$ -normal weighted incident matrix. For any non-zero vector  $\mathbf{x} := (x_1, x_2, \dots, x_n) \in \mathbb{R}_{\geq 0}^n$ , we have

$$\begin{aligned} r! \sum_{\{x_{v_1}, x_{v_2}, \dots, x_{v_r}\} \in E(H)} x_{v_1} x_{v_2} \cdots x_{v_r} &= \frac{r!}{\alpha^{\frac{1}{r}}} \sum_{e \in E(H)} \prod_{v \in e} (B^{\frac{1}{r}}(v, e) x_v) \\ &\leq \frac{r!}{\alpha^{\frac{1}{r}}} \sum_{e \in E(H)} \frac{\sum_{v \in e} (B(v, e) x_v^r)}{r} \\ &= \frac{(r-1)!}{\alpha^{\frac{1}{r}}} \|\mathbf{x}\|_r^r. \end{aligned} \quad (3)$$

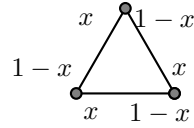
This inequality implies  $\rho(H) \leq \frac{(r-1)!}{\alpha^{\frac{1}{r}}}$ .

The equality holds if  $H$  is  $\alpha$ -normal and there is a non-zero solution  $\{x_i\}$  for the system of the following homogeneous linear equations:

$$B(v_{i_1}, e)^{1/r} \cdot x_{i_1} = B(v_{i_2}, e)^{1/r} \cdot x_{i_2} = \cdots = B(v_{i_r}, e)^{1/r} \cdot x_{i_r}, \forall e = \{x_{i_1}, x_{i_2}, \dots, x_{i_r}\} \in E(H). \quad (4)$$

Picking any vertex  $v_0$  and setting  $x_{v_0}^* = 1$ , define  $x_u^* = \left( \prod_{i=1}^l \frac{B(v_{i-1}, e_i)}{B(v_i, e_i)} \right)^{1/r}$  if there is a path  $v_0 e_1 v_1 e_2 \cdots v_l (= u)$  connecting  $v_0$  and  $u$ . Since  $H$  is connected, such path must exist. The consistent condition guarantees that  $x_u^*$  is independent of the choice of the path. It is easy to check that  $(x_1^*, \dots, x_n^*)$  is a solution of (4). Thus,  $\rho(H) = \frac{(r-1)!}{\alpha^{\frac{1}{r}}}$ .  $\square$

**Remark 1.** If  $H$  is a simple hypertree, then the “consistent” condition is automatically satisfied. In general the condition “ $H$  is  $\alpha$ -normal” doesn’t imply  $\rho(H) = (r-1)! \alpha^{-\frac{1}{r}}$ . Consider the following example  $H = C_3$ .



For any  $x \in (0, 1)$ ,  $C_3$  is  $x(1-x)$ -normal, but inconsistent unless  $x = \frac{1}{2}$ . As the consequence,  $\rho(H) = 2 \leq [x(1-x)]^{-\frac{1}{2}}$ .

Often we need compare the spectral radius with a particular value. It is convenient to introduce the following concepts.

**Definition 4.** A hypergraph  $H$  is called  $\alpha$ -subnormal if there exists a weighted incidence matrix  $B$  satisfying

1.  $\sum_{e: v \in e} B(v, e) \leq 1$ , for any  $v \in V(H)$ .
2.  $\prod_{v \in e} B(v, e) \geq \alpha$ , for any  $e \in E(H)$ .

Moreover,  $H$  is called strictly  $\alpha$ -subnormal if it is  $\alpha$ -subnormal but not  $\alpha$ -normal.

We have the following lemma.

**Lemma 4.** Let  $H$  be an  $r$ -uniform hypergraph. If  $H$  is  $\alpha$ -subnormal, then the spectral radius of  $H$  satisfies

$$\rho(H) \leq (r-1)! \alpha^{-\frac{1}{r}}.$$

Moreover, if  $H$  is strictly  $\alpha$ -subnormal then  $\rho(H) < (r-1)! \alpha^{-\frac{1}{r}}$ .

*Proof.* The proof is similar to inequality (3). For any non-zero vector  $\mathbf{x} := (x_1, x_2, \dots, x_n) \in \mathbb{R}_{\geq 0}^n$ , we have

$$\begin{aligned} r! \sum_{\{x_{v_1}, x_{v_2}, \dots, x_{v_r}\} \in E(H)} x_{v_1} x_{v_2} \cdots x_{v_r} &\leq \frac{r!}{\alpha^{\frac{1}{r}}} \sum_{e \in E(H)} \prod_{v \in e} (B^{\frac{1}{r}}(v, e) x_v) \\ &\leq \frac{r!}{\alpha^{\frac{1}{r}}} \sum_{e \in E(H)} \frac{\sum_{v \in e} (B(v, e) x_v^r)}{r} \\ &\leq \frac{(r-1)!}{\alpha^{\frac{1}{r}}} \|\mathbf{x}\|_r^r. \end{aligned}$$

This inequality implies  $\rho(H) \leq \frac{(r-1)!}{\alpha^{\frac{1}{r}}}$ . When  $H$  is strictly  $\alpha$ -subnormal, this inequality is strict, and thus  $\rho(H) < \frac{(r-1)!}{\alpha^{\frac{1}{r}}}$ .  $\square$

**Definition 5.** A hypergraph  $H$  is called  $\alpha$ -supernormal if there exists a weighted incidence matrix  $B$  satisfying

1.  $\sum_{e: v \in e} B(v, e) \geq 1$ , for any  $v \in V(H)$ .
2.  $\prod_{v \in e} B(v, e) \leq \alpha$ , for any  $e \in E(H)$ .

Moreover,  $H$  is called strictly  $\alpha$ -supernormal if it is  $\alpha$ -supernormal but not  $\alpha$ -normal.

We have the following lemma.

**Lemma 5.** Let  $H$  be an  $r$ -uniform hypergraph. If  $H$  is strictly and consistently  $\alpha$ -supernormal, then the spectral radius of  $H$  satisfies

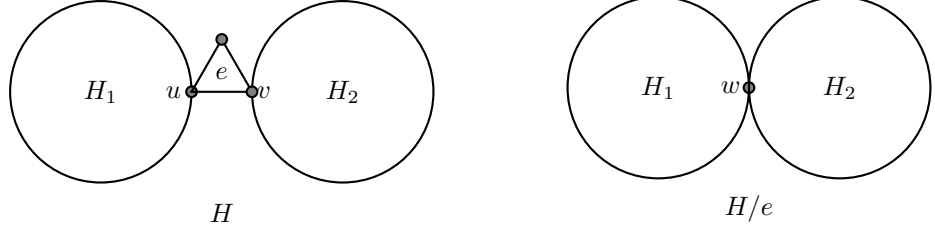
$$\rho(H) > (r-1)! \alpha^{-\frac{1}{r}}.$$

*Proof.* By the same argument as the proof of Lemma 3, the consistent condition implies that there exists a positive vector  $\mathbf{x} = (x_1^*, x_2^*, \dots, x_n^*)$  satisfying equation (4). We have

$$\begin{aligned} r! \sum_{\{x_{v_1}^*, x_{v_2}^*, \dots, x_{v_r}^*\} \in E(H)} x_{v_1}^* x_{v_2}^* \cdots x_{v_r}^* &\geq \frac{r!}{\alpha^{\frac{1}{r}}} \sum_{e \in E(H)} \prod_{v \in e} (B^{\frac{1}{r}}(v, e) x_v^*) \\ &= \frac{r!}{\alpha^{\frac{1}{r}}} \sum_{e \in E(H)} \frac{\sum_{v \in e} (B(v, e) (x_v^*)^r)}{r} \\ &\geq \frac{(r-1)!}{\alpha^{\frac{1}{r}}} \|\mathbf{x}^*\|_r^r. \end{aligned}$$

This inequality implies  $\rho(H) \geq \frac{(r-1)!}{\alpha^{\frac{1}{r}}}$ . When  $H$  is strictly  $\alpha$ -supernormal, the inequality is strict, and thus  $\rho(H) > \frac{(r-1)!}{\alpha^{\frac{1}{r}}}$ .  $\square$

By Lemma 3, an  $r$ -uniform hypergraph  $H$  has the spectral radius  $\rho_r = (r-1)! \sqrt[r]{4}$  if and only if  $H$  is consistently  $\frac{1}{4}$ -normal. In the remaining section, we only consider  $\alpha = \frac{1}{4}$ . We say an edge  $e$  is a *2-bridge* of  $H$  if  $e$  contains exactly two non-leaf vertices and  $H - e$  is disconnected. Let  $uv$  be the two non-leaf vertex of the 2-bridge edge  $e$ . The contraction, denoted by  $H/e$  is a new hypergraph obtained from  $H$  by deleting the edge  $e$  and identifying  $u$  and  $v$  into a new vertex  $w$ . In this case, we also say  $H$  is an *expansion* of  $H/e$  at  $w$ . A hypergraph  $H'$  has an expansion at  $w$  if and only if  $w$  is a cut vertex of  $H'$ , i.e.  $H = H_1 \cup H_2$  and  $H_1 \cap H_2 = \{w\}$ .



We have the following lemma.

**Lemma 6.** *Let  $H$  be an  $r$ -uniform hypergraph. Suppose that  $H$  has a 2-bridge edge  $e$ . Then we have*

1. *If  $\rho(H/e) > \rho_r$ , then  $\rho(H) > \rho_r$ .*
2. *If  $\rho(H/e) = \rho_r$ , then  $\rho(H) \geq \rho_r$ . The equality holds if and only if for any consistently  $\frac{1}{4}$ -normal weighted incidence matrix  $B$  on  $H/e$ , the sum of weights at  $w$  splits evenly, i.e.  $\sum_{e' \in E(H_1)} B(w, e') = \frac{1}{2} = \sum_{e' \in E(H_2)} B(w, e')$ .*

*Proof.* Let  $B$  be the consistently  $\alpha$ -normal weighted incident matrix associated to  $H/e$  with  $\alpha = ((r-1)!/\rho(H/e))^r$ . Let  $x := \sum_{e' \in E(H_1)} B(w, e')$ ,  $y := \sum_{e' \in E(H_2)} B(w, e') = 1 - x$ . Now we extend the matrix  $B$  to  $H$  by defining  $B(u, e) = y$ ,  $B(v, e) = x$ ,  $B(z, e) = 1$  for any leaf vertex  $z$  of  $e$ .

If  $\rho(H/e) > \rho_r$ , then  $\alpha < \frac{1}{4}$ . Observe that

$$xy \leq \frac{(x+y)^2}{4} = \frac{1}{4}.$$

Thus  $B$  is  $\frac{1}{4}$ -supernormal. Since  $H$  and  $H/e$  have the same cycle space,  $B$  is still consistent. Thus,  $\rho(H) > \rho_r$ .

If  $\rho(H/e) = \rho_r$ , then  $\alpha = \frac{1}{4}$ . If  $x = y = \frac{1}{2}$ , then  $B$  is consistently  $\frac{1}{4}$ -normal. Thus,  $\rho(H) = \rho_r$ .

If  $(x, y) \neq (\frac{1}{2}, \frac{1}{2})$ , then

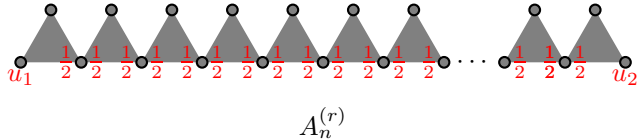
$$xy < \frac{(x+y)^2}{4} = \frac{1}{4}.$$

Thus  $B$  is  $\frac{1}{4}$ -supernormal. Thus,  $\rho(H) > \rho_r$ . □

Finally, we show that  $\rho_r$  is the limit value of the spectral radii of paths.

**Lemma 7.** *Let  $A_n^{(r)}$  be an  $r$ -uniform path with  $n$  edges, and  $\rho_r = (r-1)!\sqrt[r]{4}$ . Then, for any  $r \geq 2$ , we have  $\lim_{n \rightarrow \infty} \rho(A_n^{(r)}) = \rho_r$ .*

*Proof.* We will first show that  $\rho(A_n^{(r)}) < \rho_r$ . By labeling  $A_n^{(r)}$  as follows,



we can check that this is a strict  $\frac{1}{4}$ -subnormal labeling. Thus,  $\rho(A_n^{(r)}) < \rho_r$ . On the other hand, by the definition of  $\rho(H)$  in (1) and choosing

$$x_v^* = \begin{cases} 1 & \text{v is a leaf, } v \neq u_1, u_n; \\ y & \text{otherwise} \end{cases}$$

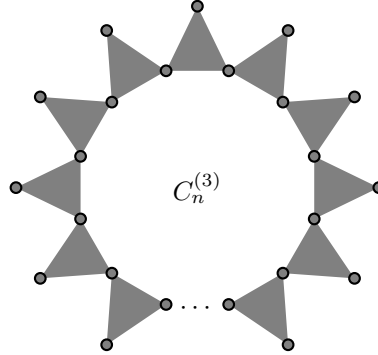
where  $y = \sqrt[r]{\frac{2n}{n+1}}$ , we have  $\rho(A_n^{(r)}) \geq \frac{P_H(x^*)}{\|x^*\|^r} = \frac{r!n \cdot y^2}{n(r-2)+(n+1)y^r} = (1 + \frac{2}{n} + \frac{1}{n^2})^{-\frac{1}{r}} \rho_r$ . Therefore,  $(1 + \frac{2}{n} + \frac{1}{n^2})^{-\frac{1}{r}} \rho_r \leq \rho(A_n^{(r)}) < \rho_r$ . By  $n \rightarrow \infty$ , we get  $\lim_{n \rightarrow \infty} \rho(A_n^{(r)}) = \rho_r$  and complete the proof of this Lemma.  $\square$

### 3 The 3-uniform hypergraphs

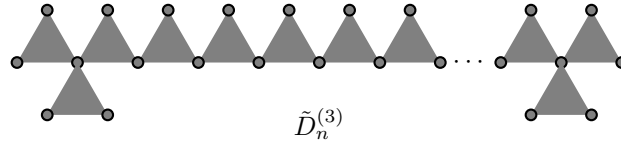
In this section, we will classify all connected 3-uniform hypergraphs with spectral radius at most  $\rho_3 := 2\sqrt[3]{4}$ . Here are our results.

**Theorem 1.** *Let  $\rho_3 = 2\sqrt[3]{4}$ . If the spectral radius of a connected 3-uniform hypergraph  $H$  is equal to  $\rho_3$ , then  $H$  must be one of the following graphs:*

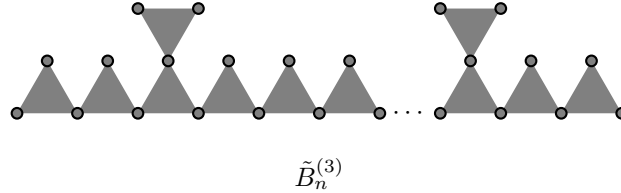
1.  $C_n^{(3)}$ : the simple cycle of  $n$  edges (for  $n \geq 3$ ).



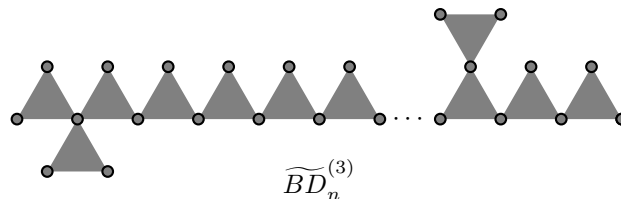
2.  $\tilde{D}_n^{(3)}$  for  $n \geq 5$ , where  $n$  is the number of edges.



3.  $\tilde{B}_n^{(3)}$  for  $n \geq 8$ , where  $n$  is the number of edges.



4.  $\widetilde{BD}_n^{(3)}$  for  $n \geq 6$ , where  $n$  is the number of edges.





5. Twelve exceptional 3-uniform hypergraphs:  $C_2^{(3)}$ ,  $S_4^{(3)}$ ,  $\tilde{E}_6^{(3)}$ ,  $\tilde{E}_7^{(3)}$ ,  $\tilde{E}_8^{(3)}$ ,  $F_{2,3,4}^{(3)}$ ,  $F_{2,2,7}^{(3)}$ ,  $F_{1,5,6}^{(3)}$ ,  $F_{1,4,8}^{(3)}$ ,  $F_{1,3,14}^{(3)}$ ,  $G_{1,1:0:1,4}^{(3)}$ , and  $G_{1,1:6:1,3}^{(3)}$ . (See Figure 1.)

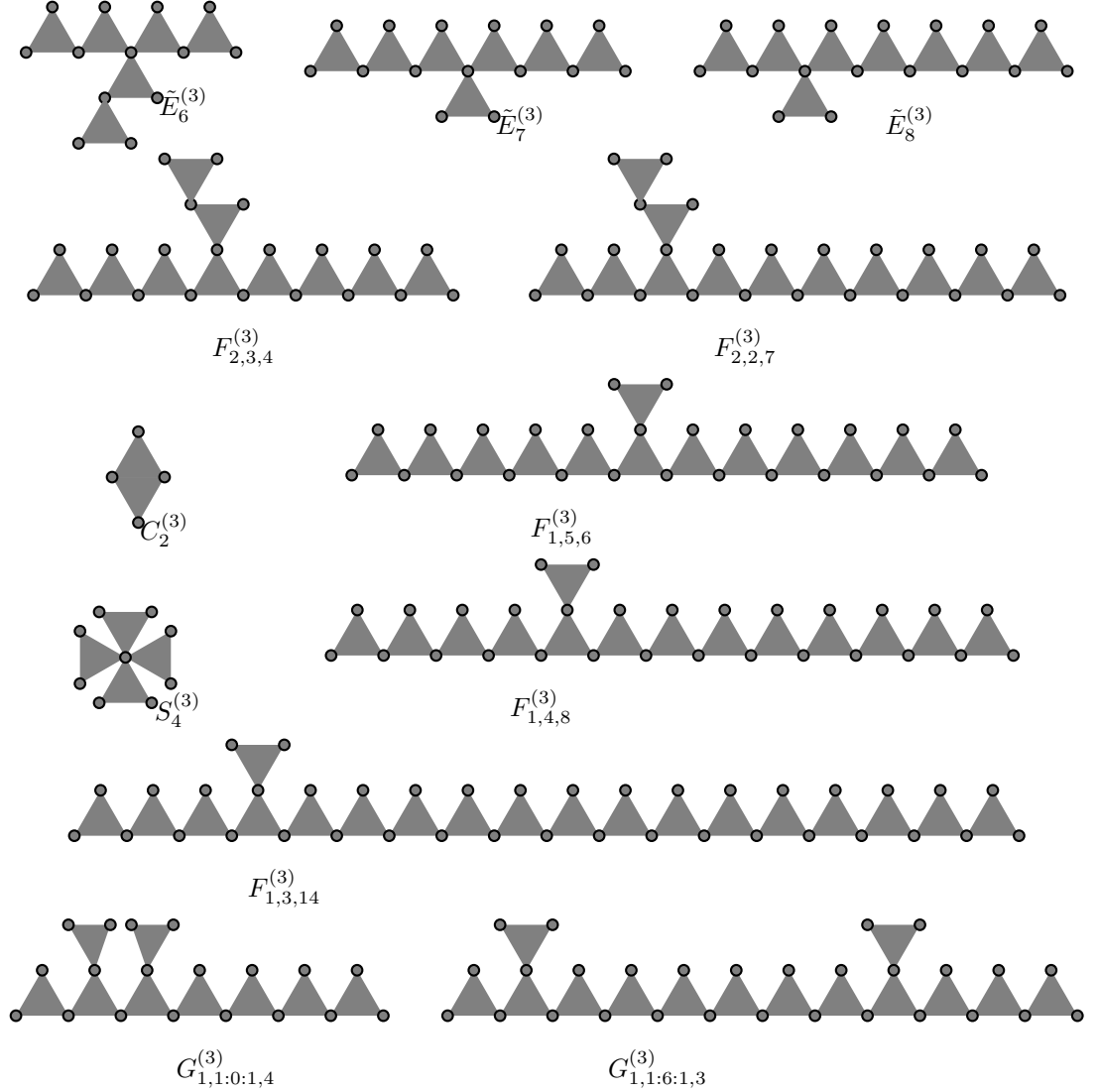
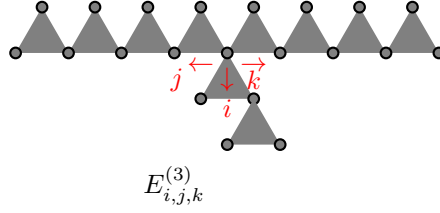
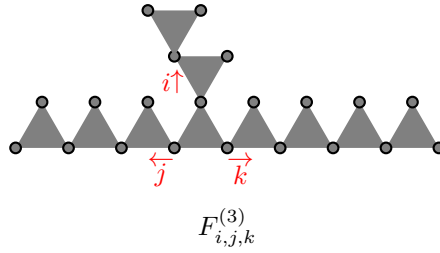


Figure 1: Twelve exceptional 3-uniform hypergraphs of spectral radius  $2\sqrt[3]{4}$ .

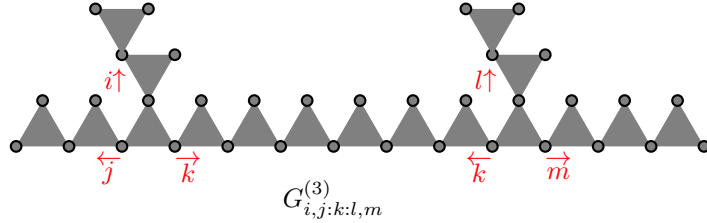
The notation of 3-uniform hypergraphs in Theorem 1 are self-defined by the figures. We denote by  $E_{i,j,k}^{(3)}$  the 3-uniform hypergraphs obtained by attaching three paths of length  $i, j, k$  to one vertex. For the consistence with  $r = 2$ , we set alias:  $E_6^{(3)} = E_{1,2,2}^{(3)}$ ,  $E_7^{(3)} = E_{1,2,3}^{(3)}$ ,  $E_8^{(3)} = E_{1,2,4}^{(3)}$ ,  $\tilde{E}_6^{(3)} = E_{2,2,2}^{(3)}$ ,  $\tilde{E}_7^{(3)} = E_{1,3,3}^{(3)}$ ,  $\tilde{E}_8^{(3)} = E_{1,2,5}^{(3)}$ , and  $D_n^{(3)} = E_{1,1,n-2}^{(3)}$ .



We denote by  $F_{i,j,k}^{(3)}$  the 3-uniform hypergraphs obtained by attaching three paths of length  $i, j, k$  to each vertex of one edge. We set alias:  $D_n^{(3)} = F_{1,1,n-3}^{(3)}$  and  $B_n^{(3)} = F_{1,2,n-4}^{(3)}$ .



We denote by  $G_{i,j:k:l,m}^{(3)}$  the 3-uniform hypergraphs obtained by attaching four paths of length  $i, j, l, m$  to four ending vertices of path of length  $k+2$  as shown in the following figure:

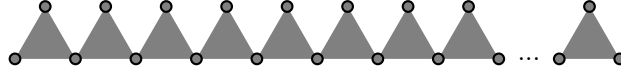


We also set alias:  $B_n^{(3)} = G_{1,1:(n-6):1,1}^{(3)}$ ,  $\bar{B}_n^{(3)} = G_{1,1:(n-7):1,2}^{(3)}$ , and  $\tilde{B}_n^{(3)} = G_{1,2:(n-8):1,2}^{(3)}$ .

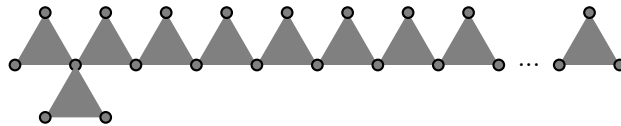
Note that any proper subgraphs of the hypergraphs listed in Theorem 1 will have the spectral radius less than  $\rho_3$ . But not all 3-uniform hypergraphs with spectral radius less than  $\rho_3$  come in this way. Here is the complete classification.

**Theorem 2.** Let  $\rho_3 = 2\sqrt[3]{4}$ . If the spectral radius of a connected 3-uniform hypergraph  $H$  is less than  $\rho_3$ , then  $H$  must be one of the following graphs:

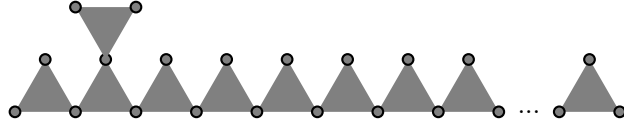
1.  $A_n^{(3)}$  for  $n \geq 1$ : a path of  $n$  edges.



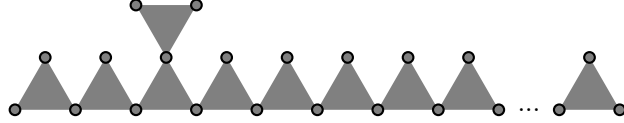
2.  $D_n^{(3)}$  for  $n \geq 3$ : where  $n$  is the number of edges.



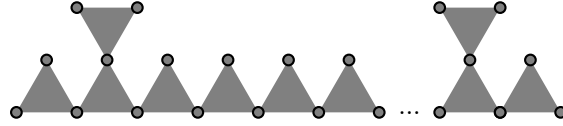
3.  $D_n^{(3)}$  for  $n \geq 4$ : where  $n$  is the number of edges.



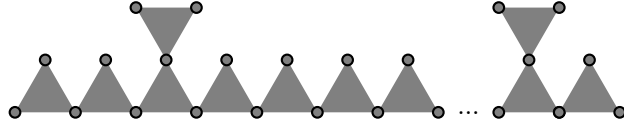
4.  $B_n^{(3)}$  for  $n \geq 5$ , where  $n$  is the number of edges.



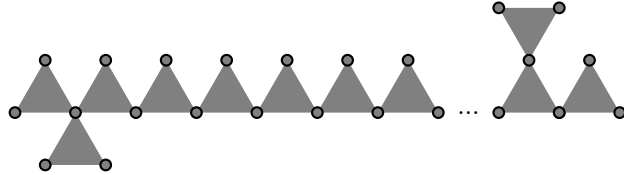
5.  $B'_n^{(3)}$  for  $n \geq 6$ , where  $n$  is the number of edges.



6.  $\bar{B}_n^{(3)}$  for  $n \geq 7$ , where  $n$  is the number of edges.

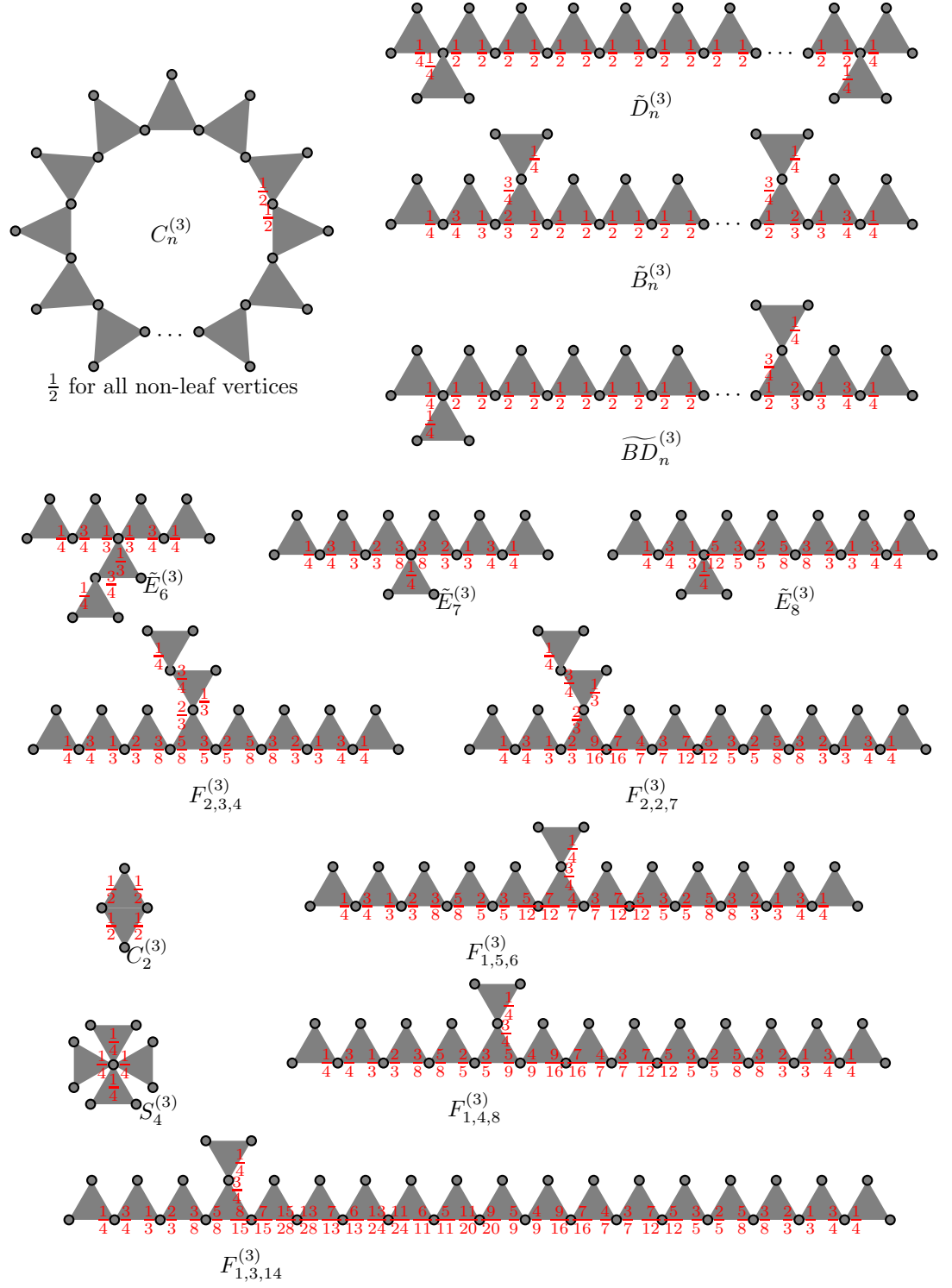


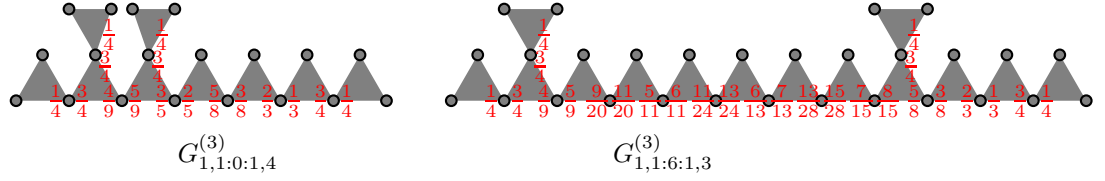
7.  $BD_n^{(3)}$  for  $n \geq 5$ , where  $n$  is the number of edges.



8. Thirty-one exceptional 3-uniform hypergraphs:  $E_6^{(3)}$ ,  $E_7^{(3)}$ ,  $E_8^{(3)}$ ,  $F_{2,3,3}^{(3)}$ ,  $F_{2,2,k}^{(3)}$  (for  $2 \leq k \leq 6$ ),  $F_{1,3,k}^{(3)}$  (for  $3 \leq k \leq 13$ ),  $F_{1,4,k}^{(3)}$  (for  $4 \leq k \leq 7$ ),  $F_{1,5,5}^{(3)}$ , and  $G_{1,1:k:1,3}^{(3)}$  (for  $0 \leq k \leq 5$ ).

*Proof of Theorem 1 and Theorem 2:* We first show that the hypergraphs listed in Theorem 1 have the spectral radius  $\rho_3$ . This is done by showing that they are all consistently  $\frac{1}{4}$ -normal. We label the value  $B(v, e)$  at vertex  $v$  near the side of edge  $e$ . If  $v$  is a leaf vertex, then it has the trivial value 1, so we will omit its labeling.





The labels show that all hypergraphs in the list of Theorem 1 are consistently  $\frac{1}{4}$ -normal and thus have the spectral radius equal to  $\rho_3$ .

We observe that the hypergraphs listed in Theorem 2 except for  $G_{1,1:k:1,3}^{(3)}$  (for  $0 \leq k \leq 5$ ) are proper subgraphs of some hypergraphs in the list of Theorem 1. By lemma 2, these hypergraphs have the spectral radius less than  $\rho_3$ . Note  $\rho(G_{1,1:6:1,3}^{(3)}) = \rho_3$ . If  $\rho(G_{1,1:k:1,3}^{(3)}) \geq \rho_3$  for some  $k \in \{0, 1, 2, 3, 4, 5\}$ , then  $\rho(G_{1,1:6:1,3}^{(3)}) \geq \rho_3$  by Lemma 6 and some labelings in  $\rho(G_{1,1:6:1,3}^{(3)})$  should be equal to  $\frac{1}{2}$ . Since this is not the case, we conclude  $\rho(G_{1,1:k:1,3}^{(3)}) < \rho_3$  for all  $k \in \{0, 1, 2, 3, 4, 5\}$ .

Now we show that the hypergraphs in Theorem 1 and 2 are the complete list of all 3-uniform hypergraphs with the spectral radius at most  $\rho_3$ . Suppose that  $H$  is a 3-uniform hypergraph with  $\rho(H) \leq \rho_3$ .

**Case 1.** If  $H$  contains  $C_2^{(3)}$  as a proper subgraph, then by lemma 2,  $\rho(H) > \rho(C_2^{(3)}) = \rho_3$ . If  $H = C_2^{(3)}$ , then  $\rho(H) = \rho_3$ . It is already in the list of Theorem 1. Thus, if  $H \neq C_2^{(3)}$ , we can assume that  $H$  is a simple hypergraph.

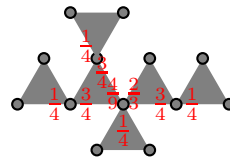
**Case 2.** If  $H$  contains a cycle,  $H$  contains  $C_n^{(3)}$  for some  $n \geq 3$ . By lemma 2,  $\rho(H) \geq \rho(C_n^{(3)}) = \rho_3$ . The equality holds if  $H = C_n^{(3)}$ , which is already in the list of Theorem 1. Thus, if  $H \neq C_n^{(3)}$ , we can assume that  $H$  is a hypertree.

**Case 3.** If there is a vertex  $v$  with degree  $d_v \geq 4$ , then  $H$  contains  $S_4^{(3)}$  as a subgraph. By lemma 2,  $\rho(H) \geq \rho(S_4^{(3)}) = \rho_3$ . The equality holds if  $H = S_4^{(3)}$ , which is already in the list of Theorem 1. Thus we can assume that every vertex in  $H$  has degree at most 3.

**Case 4.** If there exists two vertexes  $u$  and  $v$  with  $d_u = d_v = 3$ , then  $H$  contains  $\tilde{D}_n^{(3)}$  as a subgraph. By lemma 2,  $\rho(H) \geq \rho(\tilde{D}_n^{(3)}) = \rho_3$ . The equality holds if  $H = \tilde{D}_n^{(3)}$ , which is already in the list of Theorem 1. Thus we can assume that  $H$  can have at most one vertex with degree 3.

**Case 5.** Suppose that  $v$  is the unique vertex with degree 3 and all other vertices have degree at most 2. Consider the three branches attached to  $v$ .

1. If every branch has at least two edges, then  $H$  contains  $\tilde{E}_6^{(3)}$  as a subgraph. By lemma 2,  $\rho(H) \geq \rho(\tilde{E}_6^{(3)}) = \rho_3$ . The equality holds if  $H = \tilde{E}_6^{(3)}$ , which is already in the list of Theorem 1. Thus we can assume that the first branch consists of only one edge.
2. An edge  $e$  is called a branching edge if every vertex of  $e$  is not a leaf vertex. If the second branch has at least two edges and the third branch consist of a branching edge, then  $H$  consists of a subgraph  $G'$ , which can be eventually contracted to  $G$  shown below.



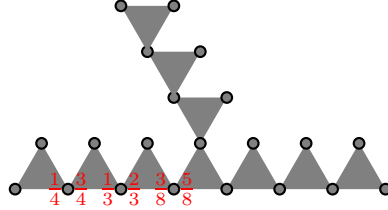
$G$  has a  $\frac{1}{4}$ -supernormal labeling

Note that the sum of the labelings of  $G$  at the center vertex is  $\frac{4}{9} + \frac{1}{3} + \frac{1}{4} > 1$ . Thus  $G$  is strictly  $\frac{1}{4}$ -supernormal and  $\rho(G) > \rho_3$ . By Lemma 2 and Lemma 6, we have  $\rho(H) > \rho(G') > \rho_3$ . Contradiction!

3. The first and second branch each consist of one edge and the third branch consists of at least one branching edge. Since  $\rho(\widetilde{BD}_n^{(3)}) = \rho_3$ ,  $H$  can not contain  $\widetilde{BD}_n^{(3)}$  as a proper subgraph. Thus the only possible hypergraphs are  $\widetilde{BD}_n^{(3)}$  and  $BD_n^{(3)}$ , which are in the list of Theorem 1 and Theorem 2 respectively.
4. There is no branching edge in  $H$ . Let  $i, j, k$  ( $i \leq j \leq k$ ) be the length of three branches of the vertex  $v$  and denote this graph by  $E_{i,j,k}^{(3)}$ . We have shown that  $i = 1$ . Note that  $E_{1,3,3}^{(3)} = \tilde{E}_7^{(3)}$  and  $E_{1,2,5}^{(3)} = \tilde{E}_8^{(3)}$  are in the list of Theorem 1. So  $(j, k)$  can only have the following choices:  $(2, 5), (2, 4), (3, 3), (2, 3), (2, 2)$  and  $(1, k), k \geq 1$ . The corresponding graphs are  $\tilde{E}_8^{(3)}, E_8^{(3)}, \tilde{E}_7^{(3)}, E_7^{(3)}, E_6^{(3)}$ , and  $D_n^{(3)}$ . These graphs are in the lists of Theorems 1 and 2.

**Case 6.** Now we can assume that all degrees of vertices in  $H$  have degrees at most 2. We will divide it into the sub-cases according to the number of branching edges.

1. If  $H$  has no branching edge, then  $H$  is a path, i.e.  $H = A_n$ , which is in the list of Theorem 2.
2. If  $H$  has exactly one branching edge, then  $H = F_{i,j,k}^{(3)}$ . We will first show that  $\rho(F_{3,3,3}^{(3)}) > \rho_3$ . We label graph  $F_{3,3,3}^{(3)}$  as follows:



By the symmetry, we only labeled one branching. Note that at the center edge, the product of weights is  $(\frac{5}{8})^3 < \frac{1}{4}$ . Thus, this is a  $\frac{1}{4}$ -supernormal labeling. Hence by lemma 5,  $\rho(F_{3,3,3}^{(3)}) > \rho_3$ . So  $H$  must not contain the subgraph  $F_{3,3,3}^{(3)}$ . Since  $i \leq j \leq k$ , we must have  $i = 1$  or  $2$ .

When  $i = 2$  and  $j = 3$ , as  $\rho(F_{2,3,4}^{(3)}) = \rho_3$ , there are only two possible hypergraphs:  $F_{2,3,3}^{(3)}$  and  $F_{2,3,4}^{(3)}$ .

When  $i = 2$  and  $j = 2$ , as  $\rho(F_{2,2,7}^{(3)}) = \rho_3$ , we must have  $2 \leq k \leq 7$ .

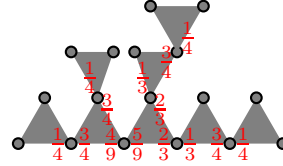
When  $i = 1$ , as  $\rho(F_{1,5,6}^{(3)}) = \rho_3$ , we must have  $j \leq 5$ . When  $j = 5$ , we have two possible hypergraphs:  $F_{1,5,5}^{(3)}$  and  $F_{1,5,6}^{(3)}$ . When  $j = 4$ , as  $\rho(F_{1,4,8}^{(3)}) = \rho_3$ , we have 5 possible hypergraphs:  $F_{1,4,k}^{(3)}$  for  $4 \leq k \leq 8$ . When  $j = 3$ , as  $\rho(F_{1,3,14}^{(3)}) = \rho_3$ , we have 12 possible hypergraphs:  $F_{1,3,k}^{(3)}$  for  $3 \leq k \leq 14$ . When  $j = 2$ , all the values of  $k$  are possible, and we get the family  $B_n^{(3)}$ . When  $j = 1$ , all the values of  $k$  are possible, and we get the family  $D_n^{(3)}$ .

*All these hypergraphs are in the list of Theorem 1 and 2.*

3. If  $H$  has exactly two branching edges, then  $H = G_{i,j:k;l,m}^{(3)}$  ( $i \leq j, l \leq m$ ).

If  $i + j \geq 3$  and  $l + m \geq 3$ , then  $H$  contains a subgraph  $G_{1,2:k:1,2}^{(3)} = \tilde{B}_{k+8}^{(3)}$ . Since the family  $\tilde{B}_n^{(3)}$  have the spectral radius equal to  $\rho_3$ , we conclude  $H$  must be  $\tilde{B}_n^{(3)}$  itself.

For the remaining cases, we can assume  $i = j = 1$ . We first show that  $\rho(G_{1,1:0:2,2}^{(3)}) > \rho_3$  (see the labeling below.)



a  $\frac{1}{4}$ -supernormal labeling of  $G_{1,1:0:2,2}^{(3)}$

Note that  $G_{1,1:k:2,2}^{(3)}$  can be obtained by expanding  $G_{1,1:0:2,2}^{(3)}$   $k$  times. By Lemma 6, we have  $\rho(G_{1,1:k:2,2}^{(3)}) > \rho_3$  for any  $k \geq 1$ . Thus, we must have  $l = 1$ . As  $\rho(G_{1,1:0:1,4}^{(3)}) = \rho_3$ , by Lemma 6, we have  $\rho(G_{1,1:k:1,4}^{(3)}) > \rho_3$  for any  $k \geq 1$ . In particular, there is no such hypergraph with  $m \geq 5$ .

If  $m = 4$ , then we only get one hypergraph  $G_{1,1:0:1,4}^{(3)}$ .

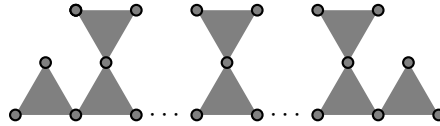
If  $m = 3$ , as  $\rho(G_{1,1:6:1,3}^{(3)}) = \rho_3$ , by Lemma 6, we get 7 hypergraphs:  $\rho(G_{1,1:k:1,3}^{(3)})$  for  $0 \leq k \leq 6$ .

If  $m = 2$ , then any  $k$  works. We get the family  $\tilde{B}_n^{(3)}$ .

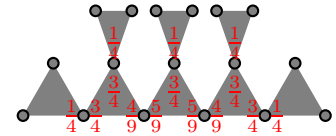
If  $m = 1$ , then any  $k$  works. We get the family  $B_n^{(3)}$ .

All these hypergraphs are in the lists of Theorems 1 and 2.

4.  $H$  contains at least three branching edges. Since all degrees of vertices are at most 2, any branching edges lie in a path. Thus,  $H$  contains a subgraph  $M'$  in the following figure. By contracting the middle edges connecting the branching edges, we get a hypergraph  $M$ . We can see that  $M$  admits the following  $\frac{1}{4}$ -supernormal labeling.



a subgraph  $M'$



after contraction:  $M$

Note that in the above labeling, the product of the center edge is  $\frac{5}{9} \cdot \frac{5}{9} \cdot \frac{3}{4} = \frac{25}{108} < \frac{1}{4}$ . So it is indeed a  $\frac{1}{4}$ -supernormal labeling. Thus,  $\rho(M) > \rho_3$ . By Lemma 6, we get  $\rho(M') > \rho_3$ . Thus,  $\rho(H) > \rho(M) > \rho_3$  by Lemma 2. Contradiction.

Therefore, all hypergraphs with spectral radius equal to  $\rho_3$  are in the list of Theorem 1, and all hypergraphs with spectral radius less than  $\rho_3$  are in the list of Theorem 2.  $\square$

## 4 General $r$ -uniform hypergraphs

For any integer  $r \geq 2$ , let  $\rho_r := (r-1)! \sqrt[r]{4}$ . In this section, we will classify all  $r$ -uniform connected hypergraphs with spectral radius at most  $\rho_r$  for all  $r \geq 4$ .

A hypergraph  $H = (V, E)$  is called *reducible* if every edge  $e$  contains at least one leaf vertex  $v_e$ . In this case, we can define an  $(r-1)$ -uniform multi-hypergraph  $H' = (V', E')$  by removing  $v_e$  from each edge  $e$ , i.e.,  $V' = V \setminus \{v_e : e \in E\}$  and  $E' = \{e - v_e : e \in E\}$ . We say that  $H'$  is *reduced* from  $H$  while  $H$  extends  $H'$ .

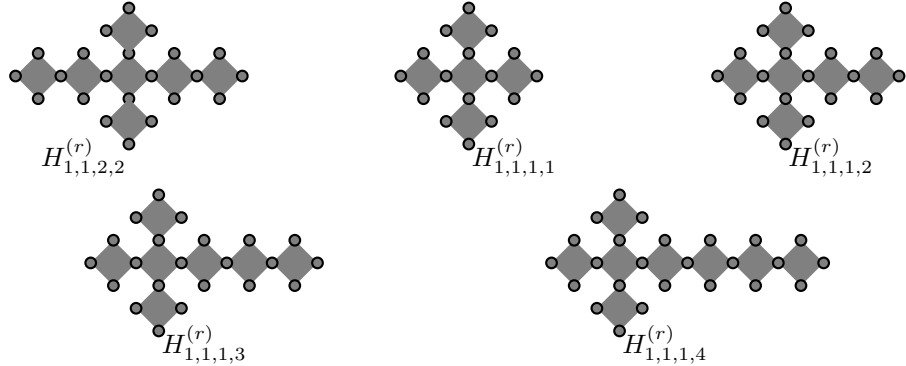
Observe that in any  $\alpha$ -normal incident matrix  $B$ , if an edge  $e$  has a leaf vertex  $v_e$ , then  $B(v_e, e) = 1$ . This leads to the following lemma.

**Lemma 8.** *If  $H$  extends  $H'$ , then  $H$  is consistently  $\alpha$ -normal if and only if  $H'$  is consistently  $\alpha$ -normal for the same value of  $\alpha$ .*

**Corollary 1.** *If  $H$  extends  $H'$ , then  $\rho(H) = \rho_r$  (or  $\rho(H) < \rho_r$ ) if and only if  $\rho(H') = \rho_{r-1}$  (or  $\rho(H') < \rho_{r-1}$ ).*

We will use the similar notion for those special  $r$ -uniform hypergraphs with spectral radius at most  $\rho_r$ . For  $r = 2$ , by Smith's theorem, the graph with spectral radius less than 2 are  $A_n, D_n, E_6, E_7, E_8$ ; the graph with spectral radius equal to 2 are  $C_n = (\tilde{A}_n), \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ , and  $\tilde{E}_8$ . For any  $r \geq 3$ , let  $A_n^{(r)}, D_n^{(r)}, E_6^{(r)}, E_7^{(r)}, E_8^{(r)}, C_n^{(r)}, \tilde{D}_n^{(r)}, \tilde{E}_6^{(r)}, \tilde{E}_7^{(r)}$ , and  $\tilde{E}_8^{(r)}$  denote the  $r$ -uniform hypergraphs extending from the graphs of Smith's list by  $r-2$  times. We can extend the graphs in Theorems 1 and 2 in a similar way. Are there any new hypergraphs not extended from the list of smaller  $r$ ?

**Theorem 3.** *For  $r \geq 5$ , every  $r$ -uniform hypergraphs with spectral radius at most  $\rho_r$  is reducible. For  $r = 4$ , irreducible hypergraphs with spectral radius at most  $\rho_r$  are the following hypergraphs.*



*Proof.* Let  $H$  be an  $r$ -uniform hypergraph with  $\rho(H) \leq \rho_r$ .

1. If  $H$  is not simple, then  $H$  contains a subgraph that consists of two edges intersecting on  $s \geq 2$  vertices. Call this subgraph  $G_s^{(r)}$ . Define a weighted incident matrix  $B$  of  $G_s^{(r)}$  as follows: for any vertex  $v$  and edge  $e$  (called the other edge  $e'$ ),

$$B(v, e) = \begin{cases} \frac{1}{2} & \text{if } v \in e \cap e', \\ 1 & \text{if } v \in e \setminus e', \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that  $B$  is consistently  $\frac{1}{4}$ -supernormal. It is strict if  $s > 2$  and  $\frac{1}{4}$ -normal if  $s = 2$ . We have

$$\rho(H) \geq \rho(G_s^{(r)}) \geq \rho_r.$$

The equality holds if and only if  $H = G_2^{(r)} = C_2^{(r)}$ . In this case,  $H$  is reducible.



2. Now assume that  $H$  is simple. If  $H$  is not a simple hypertree, then  $H$  contain a cycle. Let  $C_l = v_0 e_1 v_1 e_1 \cdots v_{l-1} e_l v_0$  be a cycle of the minimum length in  $H$ . Observe that any vertex in  $e_i$  other than  $v_{i-1}$  and  $v_i$  must be a leaf vertex in  $C_l$ . This cycle must be equal to  $C_l^{(r)}$ , which is  $\frac{1}{4}$ -normal. We have

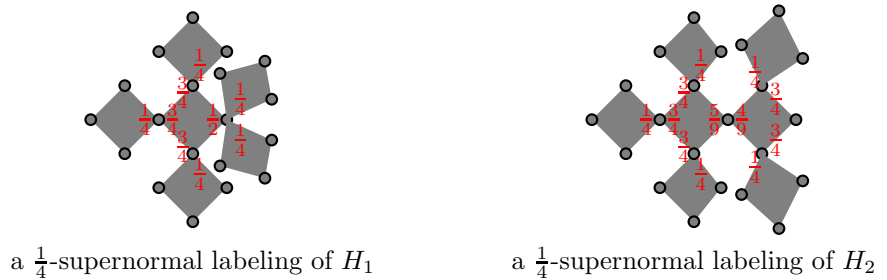
$$\rho(H) \geq \rho(C_l^{(r)}) = \rho_r.$$

The equality holds if and only if  $H = C_l^{(r)}$ . In this case,  $H$  is reducible.

3. Finally, we assume that  $H$  is a simple hypertree. Now assume that  $H$  is irreducible. There exists an edge, saying  $F_0 = \{v_1, v_2, \dots, v_r\}$  so that each vertex  $v_i$  is in another edge  $F_i$ , for  $i = 1, 2, \dots, k$ . The subgraph consisting of edges  $F_0, F_1, \dots, F_r$  is called an *edge-star*, denoted by  $S_r^{(r)}$ . Now we define  $B(v_i, F_i) = \frac{1}{4}$ ,  $B(v_i, F_0) = \frac{3}{4}$ , and  $B(v, F_i) = 1$  for each vertex  $v \neq v_i$  in  $F_i$ . Note  $\prod_{i=1}^r B(e_i, F_0) = (\frac{3}{4})^r < \frac{1}{4}$  if  $r \geq 5$ . Thus  $S_r^{(r)}$  is  $\frac{1}{4}$ -supernormal for  $r \geq 5$ . We have  $\rho(H) \geq \rho(S_r^{(r)}) > \rho_r$ . Contradiction! Thus, every  $r$ -uniform hypergraphs with spectral radius at most  $\rho_r$  is reducible.
4. It remains to consider the case  $r = 4$ . We claim that the four branches (after remove  $F_0$ ) must be all paths. Otherwise, if there is a branch containing either a branching vertex or a branching edge,  $H$  contains one of the following subgraphs  $H'_1$  and  $H'_2$ .



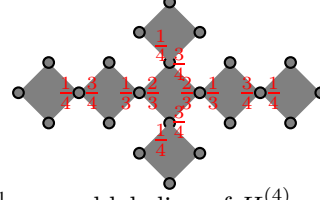
To show that  $\rho(H'_1) > \rho_4$  and  $\rho(H'_2) > \rho_4$ , it is sufficient to give a  $\frac{1}{4}$ -supernormal labeling for the contracted hypergraphs  $H_1$  and  $H_2$  as shown below.



For  $H_1$ , the product of labelings at the central edge is  $(\frac{3}{4})^3 \cdot \frac{1}{2} = \frac{27}{128} < \frac{1}{4}$ . For  $H_2$ , the product of labelings at the central edge is  $(\frac{3}{4})^3 \cdot \frac{5}{9} = \frac{15}{64} < \frac{1}{4}$ . Thus both  $H_1$  and  $H_2$  are  $\frac{1}{4}$ -supernormal. Thus for  $i = 1, 2$ ,  $\rho(H_i) > \rho_4$ , and by Lemma 6, we get  $\rho(H'_i) > \rho_4$ . Contradiction!

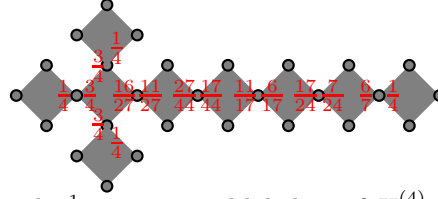
Hence, all four branches of  $F_0$  are paths. We denote  $H$  by  $H_{i,j,k,l}^{(4)}$ , where  $i, j, k$ , and  $l$  ( $i \leq j \leq k \leq l$ ) are the length of the four paths.

Note that  $\rho(H_{1,1,2,2}^{(4)}) = \rho_4$  as shown by the following  $\frac{1}{4}$ -normal labeling.



the  $\frac{1}{4}$ -normal labeling of  $H_{1,1,2,2}^{(4)}$ .

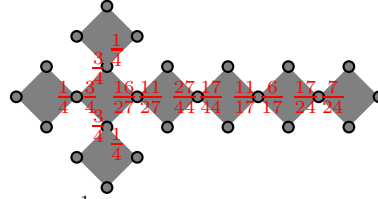
Therefore, except for  $H_{1,1,2,2}^{(4)}$ , the only possible candidates for  $H$  are  $H_{1,1,1,l}^{(4)}$ . Furthermore, if  $l = 5$ , we can label  $H_{1,1,1,5}^{(4)}$  as follows:



a strictly  $\frac{1}{4}$ -supernormal labeling of  $H_{1,1,1,5}^{(4)}$

Since  $\frac{6}{7} + \frac{1}{4} > 1$ , this is a strictly  $\frac{1}{4}$ -supernormal labeling. We get  $\rho(H_{1,1,1,5}^{(4)}) > \rho_4$ . So, by Lemma 2, we have  $\rho(H_{1,1,1,m}^{(4)}) > \rho_4$  if  $m \geq 5$ .

For  $l = 4$ , we can label  $H_{1,1,1,4}^{(4)}$  as follows:



a strictly  $\frac{1}{4}$ -subnormal labeling of  $H_{1,1,1,4}^{(4)}$

Since  $\frac{17}{24} \cdot 1 \cdot 1 \cdot 1 > \frac{1}{4}$ , this is a strictly  $\alpha$ -subnormal. So  $\rho(H_{1,1,1,4}^{(4)}) < \rho_4$ . Furthermore, by Lemma 2, we get  $\rho(H_{1,1,1,l}^{(4)}) < \rho_4$ , for all  $l = 1, 2, 3, 4$ .

Therefore, all irreducible hypergraphs with spectral radius at most  $\rho_r$  are classified in the list of Theorem 3.  $\square$

From Corollary 1, Theorem 1, Theorem 2 and Theorem 3, we have the following theorems.

**Theorem 4.** Let  $r \geq 4$  and  $\rho_r = (r-1)!\sqrt[r]{4}$ . If the spectral radius of a connected  $r$ -uniform hypergraph  $H$  is less than  $\rho_r$ , then  $H$  must be one of the following graphs:

1.  $A_n^{(r)}$ ,  $D_n^{(r)}$ ,  $D_n'^{(r)}$ ,  $B_n^{(r)}$ ,  $B_n'^{(r)}$ ,  $\bar{B}_n^{(r)}$ ,  $BD_n^{(r)}$ ,  $E_6^{(r)}$ ,  $E_7^{(r)}$ ,  $E_8^{(r)}$ ,  $F_{2,3,3}^{(r)}$ ,  $F_{2,2,j}^{(r)}$  (for  $2 \leq j \leq 6$ ),  $F_{1,3,j}^{(r)}$  (for  $3 \leq j \leq 13$ ),  $F_{1,4,j}^{(r)}$  (for  $4 \leq j \leq 7$ ),  $F_{1,5,5}^{(r)}$ , and  $G_{1,1;j:1,3}^{(r)}$  (for  $0 \leq j \leq 5$ ). Those are the  $r$ -uniform hypergraphs extending from the hypergraphs in the list of Theorem 2 by  $r-3$  times.

2.  $H_{1,1,1,1}^{(r)}$ ,  $H_{1,1,1,2}^{(r)}$ ,  $H_{1,1,1,3}^{(r)}$ ,  $H_{1,1,1,4}^{(r)}$ . Those are the  $r$ -uniform hypergraphs extending from the hypergraphs from the list of Theorem 3 by  $r - 4$  times.

**Theorem 5.** Let  $r \geq 4$  and  $\rho_r = (r - 1)! \sqrt[r]{4}$ . If the spectral radius of a connected  $r$ -uniform hypergraph  $H$  is equal to  $\rho_r$ , then  $H$  must be one of the following graphs:

1.  $C_n^{(r)}$ ,  $\tilde{D}_n^{(r)}$ ,  $\tilde{B}_n^{(r)}$ ,  $\widetilde{BD}_n^{(r)}$ ,  $C_2^{(r)}$ ,  $S_4^{(r)}$ ,  $\tilde{E}_6^{(r)}$ ,  $\tilde{E}_7^{(r)}$ ,  $\tilde{E}_8^{(r)}$ ,  $F_{2,3,4}^{(r)}$ ,  $F_{2,2,7}^{(r)}$ ,  $F_{1,5,6}^{(r)}$ ,  $F_{1,4,8}^{(r)}$ ,  $F_{1,3,14}^{(r)}$ ,  $G_{1,1:0:1,4}^{(r)}$ , and  $G_{1,1:6:1,3}^{(r)}$ . Those are the  $r$ -uniform hypergraphs extending from the hypergraphs in the list of Theorem 1 by  $r - 3$  times.
2.  $H_{1,1,2,2}^{(r)}$ , which extends  $r - 4$  times from the hypergraph  $H_{1,1,2,2}^{(4)}$  in Theorem 3.

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